MEMORANDUM
RM-3540-PR
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ON THE NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATION $\dot{x}(t) = g_1[x(t)]\, g_2[x(t-1)]$

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. The mathematical research presented here concerns the periodic solutions of differential-difference equations, which arise in a wide variety of control problems.

SUMMARY

In this Memorandum the author studies the asymptotic behavior of solutions of real difference-differential equations, and the question of the existence and behavior of periodic solutions. The equations considered arise in a wide variety of control problems.

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CONTENTS

ON THE NONLINEAR DIFFERENCE–DIFFERENTIAL EQUATION $\dot{x}(t) = g_1[x(t)]g_2[x(t-1)]$

1. INTRODUCTION

Real difference–differential equations of the form

$$(1) \qquad \dot{x}(t) = g_1[x(t)]\, g_2[x(t-1)]$$

have been discussed, for linear g_1 and g_2, by E. M. Wright [7], S. Kakutani and L. Markus [4], and G. S. Jones [3].

In this Memorandum we shall prove some results on the asymptotic behavior of solutions, and on the existence and behavior of periodic solutions, assuming only that g_2 is continuous and g_1 satisfies a Lipschitz condition, i.e., that there exists a number L such that, for any pair of real numbers x_1 and x_2, $|g_1(x_1) - g_1(x_2)| < L\,|x_1 - x_2|$.

It can be shown by means of the theory of contingencies [5], or by means of obvious adaptations from standard techniques of ordinary differential equations [2], that for any function $\varphi(t)$ in the set $C[0, 1]$ of continuous real–valued functions defined on the closed unit interval, there exists a unique continuous function $x_\varphi(t)$ defined for all $t \geq 0$ such that

$$x_\varphi(t) = \varphi(t) \quad \text{for} \quad 0 \leq t \leq 1,$$

$$\dot{x}_\varphi(t) = g_1[x_\varphi(t)]\, g_2[x_\varphi(t-1)] \quad \text{for} \quad t > 1.$$

Any such function $x_{\varphi}(t)$ is called a "solution curve" to (1).

2. CONSTANT SOLUTIONS

The constant solutions to (1) divide into two classes: those that are zeros of g_1, and those that are zeros of g_2 but not of g_1. The former we shall call "current critical points," and the latter we shall call "delayed critical points." In view of the Lipschitz condition on g_1, current critical points cannot be crossed by any solution curves for $t > 1$, and thus they divide solution curves into disjoint, nonintersecting families for $t > 1$.

We say that a constant solution c is "stable" if there exists an $\epsilon > 0$ such that if $\| \varphi - c \| < \epsilon$, then $\lim_{t \to \infty} x_{\varphi}(t) = c$.

It is often easy to determine whether or not a constant solution is stable. We shall call a critical point c "attractive" if, for x sufficiently close to c, the sign of $g_1(x)\, g_2(x)$ is opposite to the sign of $x - c$; and we shall call c "repulsive" if, for x sufficiently close to c, the sign of $g_1(x)\, g_2(x)$ is the same as the sign of $x - c$. Otherwise, we shall call a critical point "attractive–repulsive." The facts are best summarized in tabular form. The entries for current critical points are obvious, but those for delayed critical points are a bit tricky. The reader may well wonder why we are able to say only that repulsive and repulsive–attractive delayed critical points are not "necessarily" stable, in view of the fact that

Table

CRITERIA FOR STABILITY OF A CONSTANT SOLUTION

Solution	Current Critical Point	Delayed Critical Point
Repulsive	Not stable	Not necessarily stable
Repulsive—attractive	Not stable	Not necessarily stable
Attractive	Stable	Stable if $\lvert f(c)g'(c) \rvert < \pi/2$, otherwise not necessarily stable

one can always find constant functions in the neighborhood of such critical points for which the corresponding solution will, in the short run, tend away from the critical point. The answer, of course, is that in general we have no guarantee that in the long run this solution will not return to c and approach arbitrarily close to it (with damped oscillations in the repulsive case, or from the attractive side in the attractive—repulsive case). The author has been unable to construct an example in which this actually happens, however, and so we must consider the question of whether or not a delayed repulsive or repulsive—attractive critical point can be stable as unsolved. It is obvious, of course, that unstable critical points of these sorts exist.

The fact that an attractive delayed critical point is stable if $|g_1(c)\, g_2'(c)| < \pi/2$ can be viewed as a special case of a theorem of Wright [6]. The method is simply to consider the equation as being a perturbed linear equation in a suitably small neighborhood of the critical point. The fact that the critical point need not be stable if $|g_1(c)\, g_2'(c)| \geq \pi/2$ is shown by the example

$$\dot{x}(t) = -\frac{\pi}{2}\, x(t-1),$$

to which the function

$$x(t) = k \cos \frac{\pi}{2} t$$

is a solution for any value of k.

3. OTHER PERIODIC SOLUTIONS

Now let us turn our attention to nonconstant periodic solutions of (1). Since current critical points divide solution curves into non-intersecting families, it is clear that any nonconstant periodic solution $x_\varphi(t)$ must take values confined to an interval (or a half-line) in which $g_1(x)$ does not change sign. Since any periodic solution must have maxima and minima (at which $\dot{x}(t) = 0$), it follows that any nonconstant periodic solution must pass through at least one delayed critical point.

It is convenient to restrict our attention to "really isolated" delayed critical points. By a really isolated delayed critical point c we mean a point such that, if c' is any other delayed critical point, then there is a current critical point between c and c'. The important feature about really isolated delayed critical points for our purposes is the fact that a nonconstant periodic solution that intersects a really isolated delayed critical point cannot intersect any other critical points.

In the sequence of lemmas which follows, we shall assume, unless the contrary is stated, that c is a really isolated delayed critical point, and that $x(t)$ is a nonconstant periodic solution that intersects it.

Lemma 1. The critical point c is not repulsive-attractive.

Proof. If c were repulsive—attractive, then $x(t)$ would either be monotone nonincreasing or monotone nondecreasing, and thus could not be nontrivially periodic. Thus c is either repulsive or attractive.

Lemma 2. Between any two successive intersections of $x(t)$ with $x = c$, there is exactly one extreme point (relative minimum or maximum), and between any two successive extreme points there is exactly one intersection of $x(t)$ with $x = c$.

Proof. Let τ be the primitive period of $x(t)$; i.e., let τ be the least positive value such that $x(t + \tau) = x(t)$ for all t. Then the map $t \to t + 1 \pmod{\tau}$ will carry the points of intersection of $x(t)$ with $x = c$ for $0 \leq t < \tau$ onto the points in the same interval at which $\dot{x}(t) = 0$. Thus the number of extreme points in a period is at most equal to the number of intersections with c. It is clear that there must be at least one extreme point between two successive intersections with c, and thus that there must be exactly one extreme point, for if there were more, then we would have more extreme points than intersections with c. Thus every point where $\dot{x}(t) = 0$ is a relative maximum or a relative minimum. Note that this argument holds even if we interpret "point of intersection with c" and "extreme point" as meaning "closed interval on which $x(t) = c$" and "closed interval on which $\dot{x}(t) = 0$." Since, however,

the above argument shows that there is no point at which both $x(t) = c$ and $\dot{x}(t) = 0$, this latter situation cannot arise. This observation is worth putting in a separate lemma.

Lemma 3. The curve $x(t)$ *is never tangent to* $x = c$.

Now let us turn to the question of how regularly $x(t)$ intersects $x = c$. Define the *crossing number* of $x(t)$ to be the least number n such that $x(t)$ intersects $x = c$ exactly n times in some closed unit interval of time.

Lemma 4. If the crossing number of $x(t)$ *is* n, *then for any positive* t_o, $x(t)$ *intersects* $x = c$ *either* n *or* $n + 1$ *times in the interval* $t_o \leq t \leq t_o + 1$.

Proof. By definition, $x(t)$ intersects $x = c$ at least n times in any closed unit interval. Let t_o be chosen so that $x(t) - c = 0$ exactly n times in the interval $t_o \leq t \leq t_o + 1$. Let k be some quantity greater than 1, and let n'' denote the number of zeros of $x(t) - c$ in the interval $t_o + k \leq t \leq t_o + k + 1$. Let n' denote the number of zeros of $x(t) - c$ in the interval $t_o + 1 < t < t_o + k$. Let e, e', and e'' denote the number of extreme points for $x(t)$ in the intervals $t_o \leq t < t_o + 1$, $t_o + 1 \leq t \leq t_o + k$, and $t_o + k < t < t_o + k + 1$, respectively.

It is obvious that $n + n' = e' + e''$. Now there are two possibilities to consider: Either the first "interesting" point in $t_o \le t \le t_o + k + 1$ is an extremum or it is a zero of $x(t) - c$. In the former case, we have the following obvious inequalities:

$$n + n' + n'' \le e + e' + 2'' ,$$
$$n \ge e - 1 .$$

Using the fact that $n + n' = e' + e''$, we derive

$$e' + e'' + n'' \le e + e' + e'' ,$$
$$n'' \le e \le n + 1 .$$

In the latter case, we have

$$n + n' + n'' \le e + e' + e'' + 1 ,$$
$$n \ge e .$$

We derive

$$e' + e'' + n'' \le e + e' + e'' + 1 ,$$
$$n'' \le e + 1 \le n + 1 .$$

Thus in either case we have $n \le n'' \le n + 1$, which completes the proof.

In other words, Lemma 4 says that the number of zeros of $x(t) - c$ per unit time is nearly independent of t.

The proof of Lemma 4 does not depend strongly on the periodicity of $x(t)$. If we delete the parts of the argument that do depend on periodicity, we are left with the following lemma.

Lemma 5. Let $y(t)$ *be any solution, and let* $y(t) - c$ *have* n *zeros in the interval* $t_o \leq t \leq t_o + 1$. *Then* $y(t) - c$ *has at most* $n + 1$ *zeros in the interval* $t_o + k \leq t \leq t_o + k + 1$, *for any* $k \geq 0$.

Proof. For $k \geq 1$, the proof is so similar to the periodic case that we shall leave it to the reader. For $k < 1$, let n' denote the number of zeros in $t_o \leq t < t_o + k$, n'' denote the number in $t_o + k \leq t \leq t_o + 1$, and n''' denote the number in $t_o + 1 < t \leq t_o + k + 1$. Let e' denote the number of extreme points in $t_o \leq t < t_o + k$, e'' the number in $t_o + k \leq t < t_o + 1$, and e''' the number in $t_o + 1 \leq t < t_o + k + 1$. Clearly we have

$$n''' - 1 \leq e''',$$

$$n' = e'''.$$

Thus we get

$$n' + n'' = e''' + n'' \geq n''' + n'' - 1,$$

which was what we sought to prove.

Lemma 6. *If* $x(t)$ *has crossing number* n, *then there exists open unit intervals in which* $x(t)$ *has* $n+1$ *zeros.*

Proof. Let t_o be a zero of $x(t) - c$. Then, by Lemma 3, $t_o + 1$ cannot be a zero. Hence, by continuity, we can find an $\epsilon > 0$ such that there is no zero of $x(t) - c$ in the interval $t_o + 1 - \epsilon \leq t \leq t_o + 1 + \epsilon$. Thus if there are only n zeros in the interval $t_o \leq t \leq t_o + 1$, there will be $n - 1$ zeros in the interval $t_o + \epsilon \leq t \leq t_o + 1 + \epsilon$, which is impossible by hypothesis. It follows that there are $n + 1$ zeros in the interval $t_o \leq t \leq t_o + 1$, and thus there are $n + 1$ zeros in the open interval $t_o - \epsilon < t < t_o + 1 - \epsilon$.

To combine Lemma 5 and Lemma 6 in an attractive package, we need another very simple lemma.

Lemma 7. *If a continuous function has* $n + 1$ *zeros in an open unit interval, at each of which it changes sign, then there is an* $\epsilon > 0$ *such that any continuous function within* ϵ *(in the uniform metric) of the function in question will have at least* $n + 1$ *zeros in the same open unit interval.*

Proof. Let $y(t)$ be the function in question, and let $t_o < t < t_o + 1$ be the open unit interval. Let t_i $(i = 1, 2, \ldots, n+1)$ be the zeros in the interval. Let z_i $(i = 0, 1, \ldots, n+1)$ be points

chosen so that $t_i < z_i < t_{i+1}$. Thus the nonzero quantities $y(z_o)$, $y(z_1)$, ..., are alternately positive and negative. If we choose

$$\epsilon = \frac{1}{2} \min_{0 \le i \le n+1} |y(z_i)|,$$

then any continuous function within ϵ of y will also be alternately positive and negative at the z_i's, and thus will have at least $n + 1$ zeros in the interval.

The following theorem simply combines the results of Lemmas 5, 6, and 7.

Theorem 1. *Given an equation of the form (1), let* c *be a delayed critical point such that if* c' *is any other delayed critical point then there is a current critical point between them. Let* $x(t)$ *be a nonconstant periodic solution with crossing number* n *with respect to* c. *Then any function* $\varphi(t) \in C[0, 1]$ *that does not cross any current critical points, and is such that* $x_\varphi(t)$ *is asymptotic to* $x(t)$, *must cross* c *at least* n *times.*

Example. Consider

$$\dot{x}(t) = \alpha_k (-1)^{k+1} x(t-1),$$

where

$$\alpha_k = (k\pi + \frac{\pi}{2}), \quad k \text{ an integer}.$$

Then $\cos(\alpha_k t)$ is a nonconstant periodic solution, and $b = 0$ is an associated really isolated delayed critical point. Note that $\cos(\alpha_k t)$ has zeros $\left|\frac{2}{2k+1}\right|$ units apart, and thus has frequency of oscillation $|k|$. Theorem 1 now says, for example, that the solution to

$$\dot{x}(t) = \frac{7\pi}{2}\, x\,(t-1),$$

which corresponds to

$$\varphi(t) = \cos 2\pi t, \quad 0 \leq t \leq 1,$$

cannot approach any of the nonconstant periodic solutions $\cos(\alpha_3 t + \beta)$, since the latter have crossing numbers three while the former has only two zeros.

Note that in the example above the c is repulsive for odd positive k and attractive for even positive k. This situation is typical, as the following lemma shows.

Lemma 8. _Let_ c _be a really isolated delayed critical point with an associated nonconstant periodic solution_ $x(t)$ _having crossing number_ n. _Then_ c _is repulsive if and only if_ n _is odd, and attractive if and only if_ n _is even._

Proof. Suppose $x(t)$ has odd crossing number n. Let $t_o \leq t \leq t_o + 1$ be an interval in which $x(t) - c$ has n zeros.

Assume, for definiteness, that $x(t_o) > c$. Then since n is odd, it follows that $x(t_o + 1) < c$. If c is attractive, then $\dot{x}(t_o + 1) < 0$, and if t_1 is the first zero in the interval, then $\dot{x}(t) < 0$ for $t_o + 1 \leq t \leq t_1 + 1$. Thus if ϵ is a sufficiently small quantity, there can be only $n-1$ zeros in the interval $t_1 + \epsilon \leq t \leq t_1 + 1 + \epsilon$. This is a contradiction, so c cannot be attractive, and thus (by Lemma 1) must be repulsive. An entirely similar argument shows that if n is even, c cannot be repulsive, and thus our lemma is proved.

Note that in the example above the periodic solution $x = \cos(\alpha_k t)$ has period $4/|2k+1|$. Thus as k gets large, the period of the periodic solution gets small. This suggests that large absolute values for $g_1(x)$ and $g_2(x)$ may be required in order to obtain short periods for periodic solutions. The following theorem confirms this suspicion.

<u>Theorem 2.</u> If $|g_1(x)| \leq k_1 \, |x-c|$, $|g_2(x)| \leq k_2$, <u>and if there exists a periodic solution oscillating about the really isolated critical point</u> c <u>with period</u> r, <u>then</u>

$$\frac{2}{r} < k_1 \, k_2 .$$

<u>Proof.</u> Let $x(t)$ be the periodic solution in question, let h be the smallest relative maximum achieved by $|x(t) - c|$, and let h be achieved between the consecutive zeros t_1 and t_2 of $x(t) - c$.

Then $t_1 + 1$ and $t_2 + 1$ are consecutive extrema of $x(t) - c$. Clearly $|x(t_2 + 1) - x(t_1 + 1)| \geq 2h$; but on the other hand,

$$|x(t_2 + 1) - x(t_1 + 1)| = \int_{t_1+1}^{t_2+1} g_1(x(s))\, g_2(x(s-1))\, ds\, |$$

$$\leq \int_{t_1}^{t_2} k_2 k_1 |x(s) - c|\, ds$$

$$< k_2 k_1 h(t_2 - t_1).$$

Thus we have

$$2h < k_2 k_1 h(t_2 - t_1).$$

Since $t_2 - t_1 < r$, it follows that

$$2 < k_1 k_2 r,$$

which completes the proof.

4. THE EXISTENCE OF NONTRIVIAL PERIODIC SOLUTIONS

The lemmas and theorems above deal with conditions that nonconstant periodic solutions must satisfy, but they shed very little light on the question as to whether or not such solutions exist. Now we shall establish some conditions that ensure the existence of at least one nonconstant periodic solution.

Theorem 3. *Let there be given an equation of the form (1), and numbers* $m > 0$, $a > 0$, $k > 0$, *and* $c_1 < c < c_2$, *such that*

(a) $g_1(c_1) = g_1(c_2) = 0$, $g_1(x) > 0$ *for* $c_1 < x < c_2$,

(b) $g_2(c) = 0$, $|g_2(x)| \geq k \min(a, |x - c|)$ *for* $c_1 < x < c_2$,

(c) $g_2(x)$ *is monotone decreasing for* $|x - c| < a/4$,

(d) $4m > g_1(x) > m \geq 4/k$ *for* $|x - c| \leq a$.

Then there exists a periodic solution $x(t)$ *to (1) with period greater than* 2 *and crossing number* 0, *such that*

$$\max[x(t) - c] \geq a, \quad \min[x(t) - c] \leq -a.$$

Proof. Define

$$M_1 = \underset{c_1<x<c_2}{\text{l.u.b.}} |g_1(x)| \qquad M_2 = \underset{c_1<x<c_2}{\text{l.u.b.}} |g_2(x)|,$$

$$L = \underset{c_1<x_1<x_2<c_2}{\text{l.u.b.}} \frac{|g_1(x_2) - g_1(x_1)|}{x_2 - x_1},$$

$$M = M_1 M_2 \qquad B = \min(c_2 - c, c - c_1).$$

Let K be the set of all functions φ in $C[0, 1]$ that satisfy all three of the following conditions:

(i) $\varphi(0) = c$,

(ii) $at \leq \varphi(t) - c \leq c_2 - Be^{-M_2L} - c$,

(iii) $0 \leq \dfrac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} \leq M$ for $0 \leq t_1 < t_2 \leq 1$.

It is easy to see that K is a closed convex family of equi–continuous functions. By the proof of the Birkhoff–Kellogg theorem [1], K has the fixed–point property. If $\varphi \in K$, then there exists a smallest $t' \geq 1$ such that $x_\varphi(t') = c + a$. It is clear that $\dot{x}_\varphi(\tau) < 0$ for $t' < \tau < t' + 1$. Now we have $x_\varphi(t' + 1) < c - a$, since otherwise we would still have

$$x_\varphi(t'+1) - x_\varphi(t') = \int_{t'}^{t'+1} \dot{x}_\varphi(\tau)\, d\tau = \int_{t'}^{t'+1} g_1(x_\varphi(\tau))\, g_2(x_\varphi(\tau - 1))\, d\tau$$

$$< \int_{t'}^{t'+1} \frac{4}{k}(-ka(\tau - t'))\, d\tau$$

$$= -2a,$$

which is a contradiction. Now there clearly exists a unique smallest $t_\varphi > 0$ such that $x_\varphi(t_\varphi) = c$. Define

$$\mathfrak{J}(\varphi) = x_\varphi(t_\varphi + t),\ 0 \leq t \leq 1,$$

and let K' be the set of all functions φ in C[0, 1] that meet all three of the following conditions:

(i') $\varphi(0) = c$,

(ii') $-a\, t \geq \varphi(t) - c \geq + Be^{-M_2 L} - c$,

(iii') $0 \geq \dfrac{\varphi(t_2) - \varphi(t_1)}{t_2 - 2_1} \geq M$ for $0 \leq t_1 < t_2 \leq 1$.

Now the map $\mathfrak{J}$ carries K into K'. If $\varphi \in K$, then $\mathfrak{J}(\varphi)$ obviously satisfies (i') and (iii'). It satisfies the second inequality in (ii') by virtue of the fact that

$$|\dot{x}_\varphi(t)| \leq M_2 L\ |x_\varphi(t) - c_1|.$$

To show that it satisfies the first inequality in (ii'), we have only to show that if $\mathfrak{J}(\varphi) = \psi$, then

(2) $\dot{\psi}(t) < -a$ if $\psi(t) > c - a$.

In view of condition (ii) on K, and condition (b), (c), and (d) in Theorem 3, we have

$$\dot{\psi}(t) < -m\ k \min (a, \ |a(t_\varphi - 1)|).$$

If $t_\varphi - 1 \geq 1/4$, then since $m \geq 4/k$ the desired inequality (2) follows trivially. Thus our consideration is reduced to the case $t_\varphi - 1 < 1/4$, $|\varphi(t_\varphi - 1) - c| < a/4$,

(3) $\dot{\psi}(t) < -m\ k\ |\varphi(t_\varphi - 1) - c|$.

How small can $\varphi(t_\varphi - 1)$ be? Suppose that $t_\varphi - 1 = h$. Then, by conditions (b) and (c) of Theorem 3, and condition (iii) on K, we have

$$a \leq h\,k\,|\varphi(t_\varphi - 1) - c|\;4m .$$

Thus we obtain

$$h\,k\,m \geq \frac{a}{4|\varphi(t_\varphi - 1) - c|} \geq \frac{a}{4\frac{a}{4}} = 1 .$$

It is also clear, however, by condition (b) on K, that

$$|\varphi(t_\varphi - 1) - c| \geq a\,h .$$

Substituting into (3), we find

$$\dot{\psi}(t) < -m\,k\,a\,h < -a ,$$

which is the inequality we sought.

Thus K_1 is carried into K_2 by $\mathfrak{T}$. <u>Mutatis mutandi</u>, K_2 is carried into K_1 by $\mathfrak{T}$. It is easy to see that $\mathfrak{T}$ is continuous on K_1 and also on K_2. Thus $\mathfrak{T}^2$ is a continuous map of K_1 into itself, and so by the Birkhoff–Kellogg theorem it has a fixed point in K_1. The solution $x_\varphi(t)$ corresponding to this fixed point is clearly a periodic solution to (1) having period greater than 2, having crossing number 0, and satisfying the inequalities

$$\max (x_\varphi(t) - c) \geq a, \quad \min (x_\varphi(t) - c) \leq -a.$$

Corollary. Let there be given an equation of the form (1), and numbers $m > 0$, $a > 0$, $k > 0$, and $c_1 < c < c_2$, such that

(a) $g_1(c_1) = g_1(c_2) = 0$, $g_1(x) < 0$ for $c_1 < x < c_2$,

(b) $g_2(c) = 0$, $|g_2(x)| \geq k \min (a, |x - c|)$ for $c_1 < x < c_2$,

(c) $g_2(x)$ is monotone increasing for $|x - c| < a/4$,

(d) $-4m < g_1(x) < -m < -4/k$ for $|x - c| \leq a$.

Then there exists a periodic solution $x(t)$ to (1) with period greater than 2 and crossing number 0, such that

$$\max [x(t) - c] \geq a \min [x(t) - c] \leq -a.$$

Proof. The proof is the same as that for Theorem 3, with signs reversed.

Corollary. Given an equation of the form (1), if c_1 and c_2 are current critical points, c is an attractive, really isolated, delayed critical point between them, g_2' exists and is continuous in a neighborhood of c, and $|g_1(c)\ g_2'(c)| > 4$, then there exists a periodic solution oscillating about c with period greater than 2 and crossing number 0, and c is unstable.

Proof. For some very small $\epsilon > 0$, take $g_2'(c) - \epsilon$ as k, and find some small $a > 0$ such that the conditions (b), (c), and (d) of Theorem 3 (or the corollary above) are satisfied. Since a may be taken as small as we please, and $\mathcal{T}^2$ carries K_1 into itself, it follows that c must be unstable.

Theorem 4. Let there be given an equation of form (1), and numbers $m > 0$, $a > 0$, $k > 0$, c_1, *and* c, *such that*

(a) $g_1(c_1) = 0$, $g_1(x) > 0$ *for* x *on the same side of* c_1 *as* c,

(b) $g_2(c) = 0$, $|g_2(x)| \geq k \min (a, |x - c|)$ *for* x *on the same side of* c_1 *as* c,

(c) $g_2(x)$ *is monotone decreasing for* $|x - c| < a/4$,

(d) $4m > g_1(x) > m > 4/2$ for $|x - c| \leq a$.

Then there exists a periodic solution $x(t)$ *to (1) with period greater than 2 and crossing number 0, such that*

$$\max (x(t) - c) \geq a, \quad \min (x(t) - c) \leq -a.$$

Proof. For definiteness, assume that $c < c_1$. Let L be a Lipschitz constant for g_1, i.e., let $|g_1(x_1) - g_1(x_2)| \leq L|x_1 - x_2|$,

and define

$$M_2 = \underset{c<x<c_1}{\text{l.u.b.}} |g_2(x)|, \qquad x_o = c_1 + (c - c_1) e^{M_2 L},$$

$$M_2^* = \underset{x_o \le x \le c_1}{\text{l.u.b.}} |g_2(x)|, \qquad M_1 = \underset{x_o \le x \le c_1}{\text{l.u.b.}} |g_1(x)|,$$

$$M = M_1 M_2^* .$$

Let K be the set of all functions φ in $C[0, 1]$ that meet all of the following conditions:

(i) $\varphi(o) = c$,

(ii) $a\, t \le \varphi(t) - c \le c_1 - (c_1 - c) e^{-M_2^* L} - c$,

(iii) $0 \le \dfrac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} \le M$ for $0 \le t_1 < t_2 \le 1$.

Similarly, define K' to be the set of all functions φ in $C[0, 1]$ that meet all of the following conditions:

(i') $\varphi(o) = c$,

(ii') $-a\, t \ge \varphi(t) - c \ge x_o - c$,

(iii') $0 \ge \dfrac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} \ge -M$ for $0 \le t_1 < t_2 \le 1$.

The rest of the proof is virtually identical to the latter portion of the proof of Theorem 3, and will not be repeated.

Corollary. Let there be given an equation of the form (1), and numbers $m > 0$, $a > 0$, $k > 0$, c_1, *and* c, *such that*

(a) $g_1(c_1) = 0$, $g_1(x) < 0$ *for* x *on the same side of* c_1 *as* c,

(b) $g_2(c) = 0$, $|g_2(x)| \geq k \min(a, |x-c|)$ *for* x *on the same side of* c_1 *as* c,

(c) $g_2(x)$ *is monotone increasing for* $|x-c| < a/4$,

(d) $4m > g_1(x) > m > 4/k$ *for* $|x-c| < a$.

Then there exists a periodic solution $x(t)$ *to (1) with period greater than 2 and crossing number 0, such that*

$$\max(x(t)-c) \geq a, \quad \min(x(t)-c) \leq -a.$$

The following are corollaries of Theorems 3 and 4 together.

Corollary. Given an equation of the form (1) that has at least one current critical point, if c *is an attractive, really isolated, delayed critical point such that* g_2' *exists and is continuous in a neighborhood of* c, *and* $|g_1(c)\, g_2'(c)| > 4$, *then there exists a periodic solution oscillating about* c *with period greater than 2 and crossing number 0, and* c *is unstable.*

The theorems and corrollaries above are easy to apply in practice. As an example, let us consider the bilinear case of (1).

That is, take $g_1(x) = x$, $g_2(x) = c - x$. Theorem 3 shows that if $c > 4$ then c is an unstable critical point, and we can find a periodic solution $x(t)$ with period greater than 2 and crossing numbers 0, such that

$$\max(x(t) - c) \geq a, \quad \min(x(t) - c) \leq -a,$$

where

$$a = \min\left(c - 4, \frac{3c}{5}\right).$$

Jones [3] has shown that in fact periodic solutions exist if $c > \pi/2$. So far as the author knows, Jones was the first to ever use fixed points of maps like $\mathfrak{J}^2$ to prove the existence of periodic solutions to differential–difference equations.

5. QUESTIONS FOR FURTHER RESEARCH

Can the constant 4 in the last corollary of Sec. 4 be replaced by $\pi/2$ in the general case as well as in the bilinear case?

Do there exist bounded solutions to (1) that are asymptotic neither to a constant solution nor to a periodic solution?

Under what circumstances do there exist periodic solutions with crossing numbers greater than 0?

REFERENCES

1. Birkhoff, G. D., and O. D. Kellogg, "Invariant Points in Function Space," Trans. Amer. Math. Soc., Vol. 23, 1922, pp. 96–115.

2. Brown, T. A., "The Asymptotic Behavior of some Nonlinear Autonomous Systems," Doctoral Dissertation, Harvard University, June 1962.

3. Jones, G. S., "The Existence of Periodic Solutions of $f'(x) = -\alpha f(x-1) \{1+f(x)\}$," J. Math. Anal. App., Vol. 5, 1962, pp. 435–450.

4. Kakutani, S. and L. Markus, "On the Nonlinear Difference–Differential Equation $y'(t) = [A - By(t-\tau)]y(t)$," Contributions to the Theory of Nonlinear Oscillations, Vol. 4, 1958, pp. 1–18.

5. Myskis, A. D., "General Theory of Differential Equations with a Retarded Argument," Translation No. 55, American Mathematical Society, New York, 1951.

6. Wright, E. M., "The Stability of Solutions of Nonlinear Difference–Differential Equations," Proc. Roy. Soc. Edin. A, Vol. 63, 1950, pp. 18–26.

7. Wright, E. M., "A Nonlinear Difference–Differential Equation," J. Reine Angew. Math., Vol. 194, 1955, pp. 66–87.